Bounds on the Achievable Region for Certain Multiple Description Coding Problems

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Abstract — An achievable region for the L-channel multiple description coding problem is presented. This region generalizes previous two-channel results of El Gamal and Cover and of Zhang and Berger. New outer bounds on the rate distortion region for memoryless Gaussian sources with mean-squared error distortion are also derived. For the Gaussian source, the achievable region meets the outer bound at certain points.

I. Problem Description

Consider a source that emits a sequence \( X^N = \{X(1), X(2), \ldots, X(N)\} \) of \( N \) independent and identically distributed (i.i.d.) random variables. \( X^N \) is encoded into \( L \) descriptions \( J_1, J_2, \ldots, J_L \) at rates \( R_1, R_2, \ldots, R_L \) nats per source symbol. Suppose that each description is either transmitted error-free or lost completely. Thus the receiver encounters one of \( 2^L \) configurations depending on which descriptions are received. Excepting the trivial case where no description is received, we can represent the receiver as a collection of \( 2^L - 1 \) decoders, where each decoder produces an output based on a non-empty subset of \( \{J_1, \ldots, J_L\} \).

Let \( L = \{1, \ldots, L\} \) and let \( 2^L \) be its power set. For every \( K \in 2^L \), let \( X^N_K = \{X^{(1)}_K, \ldots, X^{(N)}_K\} \) denote the output of the decoder whose inputs are \( \{J_k : k \in K\} \). Next let \( d_K = E[\frac{1}{2} \sum_{n=1}^N \delta_K (X^{(n)}, X^{(n)}_K)] \) denote the expected distortion per source symbol associated with the output \( X^N_K \), where \( \delta_K (\cdot, \cdot) \) is a distortion measure. Our problem is to find the set of rates \( \{R_1, \ldots, R_L\} \) and distortions \( \{d_K : K \in 2^L - \{\emptyset\}\} \) that are achievable in the usual Shannon sense. We call this region the rate-distortion (RD) region.

II. An Achievable Region

The set difference between collections of sets \( C \) and \( D \) is denoted \( C - D = \{M : M \in C : M \notin D\} \). Also, we write \( R_K \) as a shorthand for \( \sum_{K \in C} R_K \) and \( X_C \) for a collection of random variables \( \{X_N : N \in C\} \). Our first result is an achievable region for the general L-description problem.

Theorem 1 Let \( X(2^L) \) be \( 2^L \) finite-alphabet random variables jointly distributed with \( X \). Then the RD region contains the rates and distortions satisfying

\[
d_K \geq E \delta_K (X, X_K) \\
R_K \geq (|K| - 1) I(X; X_0) - H(X(2^K)) - \sum_{M \subseteq K} H(X(M)|X(2^M-(M)))
\]

for every \( K \in 2^L - \{\emptyset\} \), where \( |K| \) is the cardinality of \( K \).

In Theorem 1, \( X_0 \) is an arbitrary random variable. For \( L = 2 \), this result generalizes the result of Zhang and Berger [1]. Additionally, with \( X_0 \) set to a constant, e.g. 0, it reduces to the result of El Gamal and Cover [2].

Theorem 1 holds more generally for well-behaved continuous sources if all entropies \( H(\cdot) \) are replaced by differential entropies \( h(\cdot) \). We next focus exclusively on the Gaussian source with mean-squared error distortion.

III. The Quadratic Gaussian Case

The following outer bound generalizes a result of Ozarow [3]. For this result, a collection of \( M \) disjoint sets \( \{K_m\}_{m=1}^M \) is called a partition of a set \( K \) if \( \bigcup_{m=1}^M K_m = K \).

Theorem 2 For each \( K \in 2^L \), the achievable rates \( R_\ell, \ell \in L \), and distortions \( d_K, K \in 2^L \), satisfy

\[
e^{-2R_K} \leq \min_{(K_m)} \int_{\lambda \geq 0} (d_K \frac{1}{(d_K + \lambda)(1 + \lambda)^{M-1}}) \]

where the minimization is performed over all partitions of \( K \).

Example Let the source emit unit-variance i.i.d. Gaussian random variables, and suppose that we care only about the reconstructions from individual descriptions \( X_1, X_2, \ldots, X_L \), and the reconstruction from all the descriptions \( X_L \). An important operating point is that of equal side distortions and rates on all channels: \( d(\ell) = d \) and \( R_\ell = R \geq \frac{1}{2} \log d, \ell \in L \). For this operating point, we can show using Theorems 1 and 2, that the best central distortion achievable is

\[
d_L = \sup_{\lambda \geq 0} \left( \frac{e^{-2LR} \lambda(1 + \lambda)^{L-1}}{(d + \lambda)^L - e^{-2LR}(1 + \lambda)^{L-1}} \right).
\]

In fact, we can show that the outer and inner bounds meet everywhere on an open set in the vicinity of this point.

REFERENCES

